NEUTRON DIFFUSION IN NONMULTIPLYING MEDIA

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4/252021

INTRODUCTION

The diffusion of neutrons in a medium depends on whether the medium is multiplying, that is fissions occur in it or not. In a non-multiplying medium containing no fissile materials, the neutron diffusion equation results in a source problem that is mathematically different than the eigenvalue problem that it yields in a multiplying medium, and consequently requires a different treatment.

The source type of problem is encountered in shielding and dosimetry calculations, whereas the eigenvalue type of problem is encountered in criticality calculations.

NEUTRON ATTENUATION

We consider a beam of neutrons of initial intensity I_0 impingent on a thin target of a scattering material of thickness x and total macroscopic cross section Σ_t.

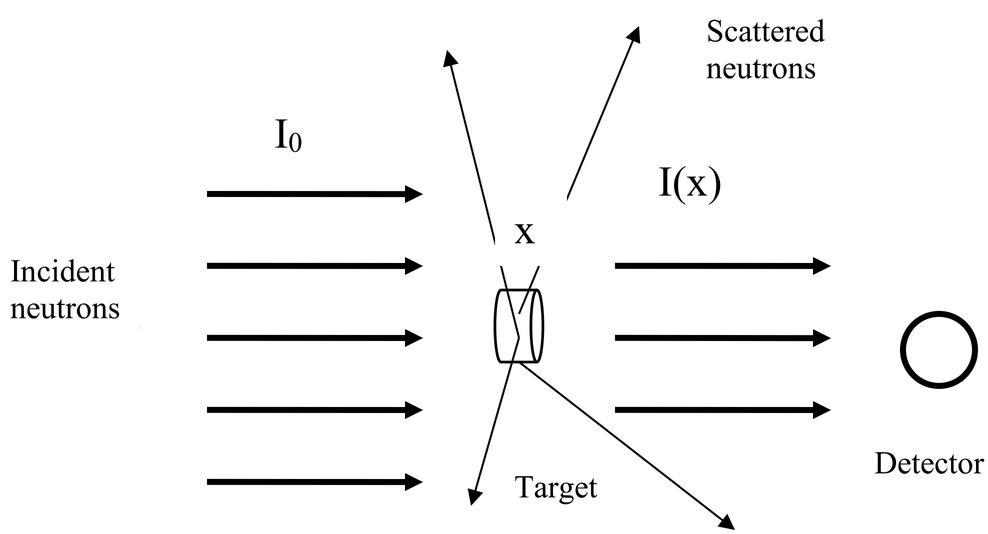

Figure 1. Thin target neutron beam attenuation.

In one dimension the particles hitting the thin target will be scattered out of the beam and only a fraction of them will reach a detector emplaced behind the target. Only the neutrons that do not interact with the target would reach the detector. This fraction reaching the detector is observed to follow an exponential attenuation law as:

$$
\begin{aligned}
I(x) &= I_0 e^{-\Sigma_t x} \\
&= I_0 e^{-\frac{x}{\lambda_t}}
\end{aligned}
\tag{1}
$$

where:

$\lambda_t = \frac{1}{\Sigma_t}$ is the total mean free path in the medium, [cm]

$\Sigma_t = N\sigma_t$ is the total cross section in the medium, [cm^{-1}]

$\text{N}=\frac{\rho \text{A}_\text{v}}{\text{M}}$ is the nuclear density of the medium [nuclei/cm^3]

ρ is the density of the medium [gm/cm^3]

A_v is Avogadro's number = 0.602×10^{24} [nuclei/mole]

M is the atomic weight [amu]

σ_t is the microscopic cross section, [1 barn =10^{-24} cm^2]

The attenuation factor in a thin shield of thickness x is:

$$\frac{I(x)}{I_0} = e^{-\Sigma_t x}$$

Taking the natural logarithm of both sides we get:

$$\Sigma_t x = -\ln \frac{I(x)}{I_0}$$

If the desired attenuation and the macroscopic cross section in the medium used as shield are known, then we can estimate the needed thickness x from:

$$x = -\frac{1}{\Sigma_t} \ln \frac{I(x)}{I_0} \tag{2}$$

If on the other hand, the thickness of the material is known, and the attenuation is measure, then the value of the microscopic cross section at a given neutron energy can be estimated from:

$$\sigma_t = -\frac{1}{Nx} \ln \frac{I(x)}{I_0} \tag{3}$$

This approach applies only to thin shields. Otherwise the concepts of build-up factors or removal cross sections can be used in thick shield and broad beam situations.

TIME DEPENDENT DIFFUSION EQUATION

The continuity equation for the neutron density n [neutrons / cm^3] is:

$$\frac{\partial n}{\partial t} = S - \Sigma_a \phi - div\overline{J} \qquad (4)$$

where:

$\overline{J}$ is the neutron current

$\text{div}\ \overline{\text{J}} = \nabla.\overline{\text{J}} = \frac{\partial \text{J}_x}{\partial \text{x}} + \frac{\partial \text{J}_y}{\partial \text{y}} + \frac{\partial \text{J}_z}{\partial \text{z}}$, is the divergence operator

S is the source strength

ϕ is the neutron flux [neutrons/(cm^2.sec)]

Σ_a is the macroscopic absorption cross section [cm^{-1}]

Notice that the divergence operator acts on the components of a vector such as the current $\overline{J}$ and generates a scalar quantity.

The neutron current is described by Fick's law as:

$$\overline{J} = -Dgrad\phi = -D\nabla\phi \qquad (5)$$

where:

D is the diffusion coefficient [cm]

$\text{grad}\ \phi = \nabla\phi = \frac{\partial \phi}{\partial \text{x}}\hat{i} + \frac{\partial \phi}{\partial \text{y}}\hat{j} + \frac{\partial \phi}{\partial \text{z}}\hat{k}$

$\hat{i}, \hat{j}, \hat{k}$ are the cartesian unit vectors in the x, y and z directions

It must be observed here that taking the gradient of a scalar function such as the flux ϕ, generates a vector as the current $\overline{J}$.

Combining Eqns. 4 and 5 we obtain the diffusion equation for the neutron flux:

$$\begin{aligned}\frac{1}{v}\frac{\partial \phi(x,y,z,t)}{\partial t} &= S - \Sigma_a \phi(x,y,z,t) + \text{div}\,(\text{Dgrad}\,\phi(x,y,z,t)) \\ &= S - \Sigma_a \phi(x,y,z,t) + \nabla.(\text{D}\nabla\phi(x,y,z,t)) \qquad (6)\end{aligned}$$

where:

$\phi = nv, n = \frac{\phi}{v}$.

STEADY STATE DIFFUSION EQUATION

The steady state diffusion equation can be obtained by equating the partial time derivative to zero:

$$+\nabla.(\mathrm{D}\nabla\,\phi(x,y,z)) - \Sigma_a\phi(x,y,z) + S = 0 \tag{7}$$

If we further assume that we have a uniform and homogeneous medium, the diffusion coefficient can be considered as a constant:

$$+D(\nabla.\nabla\,\phi(x,y,z)) - \Sigma_a\phi(x,y,z) + S = 0$$

Now the Laplacian operator is:

$$\nabla^2\phi = \nabla.\nabla\phi = \mathrm{div\,grad}\phi = \frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial \mathrm{z}}\left(\frac{\partial\phi}{\partial z}\right),$$
$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

resulting in:

$$+D\nabla^2\,\phi(x,y,z) - \Sigma_a\phi(x,y,z) + S = 0 \tag{8}$$

Dividing into the diffusion coefficient D we get:

$$\nabla^2\,\phi(x,y,z) - \frac{\Sigma_a}{D}\phi(x,y,z) = -\frac{S}{D}$$

Defining the diffusion area as:

$$L^2 = \frac{D}{\Sigma_a}[\mathrm{cm}^2] \tag{9}$$

and the diffusion length L as:

$$L = \sqrt{\frac{D}{\Sigma_a}}[\mathrm{cm}] \tag{10}$$

we can write the steady state diffusion equation as:

$$\nabla^2\,\phi(x,y,z) - \frac{1}{L^2}\phi(x,y,z) = -\frac{S}{D} \tag{11}$$

Since this is a second order partial differential equation it needs the definition of two boundary conditions for each dimension in its solution. The equation is analogous to the one for the simple harmonic oscillator, albeit it has a second order spatial time derivative rather than in time. Still, the solutions for the simple harmonic oscillator can be used by replacing the time variable by the spatial variable.

POINT NEUTRON SOURCE STREAMING IN A VACUUM

It is of interest to consider the flux generated in vacuum, such as from a nuclear reactor on a space probe, from a point source of neutrons for comparison to the case of a diffusing medium. In this case, the vacuum does not possess a diffusion coefficient or an absorption cross section and the radiation undergoes a process of "streaming" from the source. In this case, like a source of light, the decrease in the intensity is purely geometrical, and follows the inverse square law:

$$\phi(r) = \frac{S}{4\pi r^2}[n/(cm^2.\sec)] \tag{12}$$

where r is the radial distance from the source.

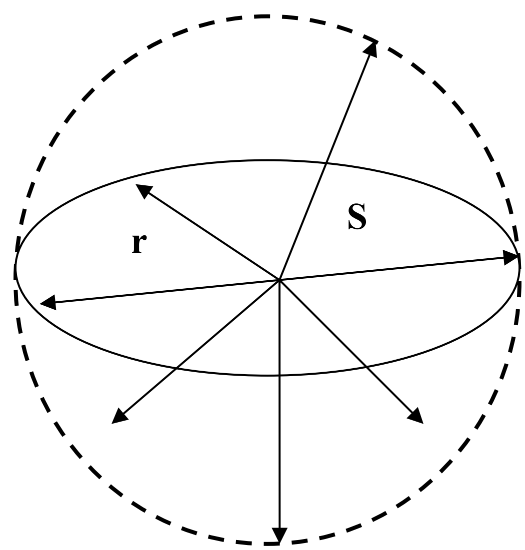

Figure 2. Particle streaming from a point source S [neutrons/sec] in a vacuum.

POINT NEUTRON SOURCE IN A DIFFUSING MEDIUM

In this case Eqn. 11 applies taken in spherical geometry as:

$$\nabla^2 \phi(r) - \frac{1}{L^2}\phi(r) = -\frac{S}{D} \tag{13}$$

Expressing the Laplacian operator in its spherical geometry form we get:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi(r)}{dr}\right) - \frac{1}{L^2}\phi(r) = -\frac{S}{D} \tag{14}$$

Notice that the partial derivative is here replaced by a total derivative since we have a single dimension in spherical geometry, which makes the equation an ordinary, rather than a partial differential equation of the second order.

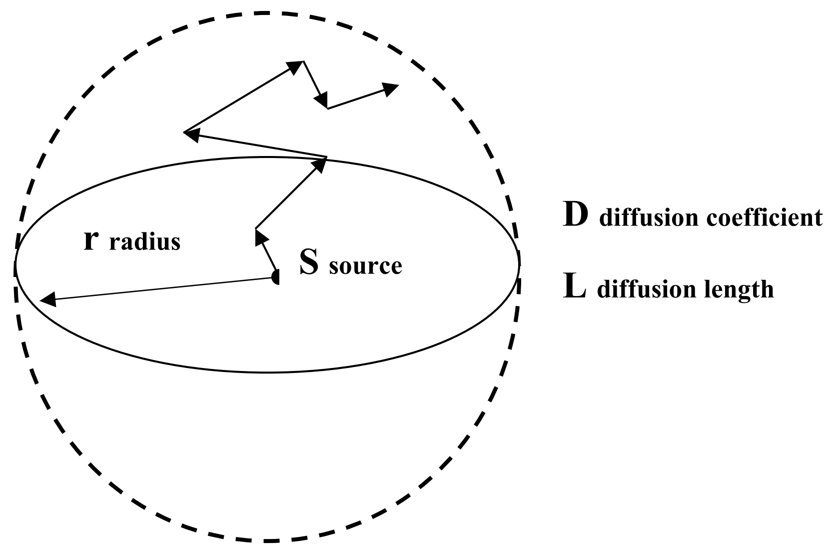

Figure 3. Particle diffusion from a point source S in a nonmultiplying medium with diffusion coefficient D and diffusion length L.

We can first consider the homogeneous part of Eqn. 14 as:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi(r)}{dr}\right) - \frac{1}{L^2}\phi(r) = 0 \tag{15}$$

To solve such a spherical geometry ordinary differential equation it is convenient to make the change of variable:

$$w(r) = r\phi(r) \Rightarrow \phi(r) = \frac{w(r)}{r} \tag{16}$$

Recalling the chain rule of differentiation:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{17}$$

and applying it to Eqn. 16, we get:

$$\frac{d\phi}{dr} = \frac{d}{dr}(w.\frac{1}{r}) = +w(-\frac{1}{r^2}) + \frac{1}{r}\frac{dw}{dr}$$

Multiplying both sides by r^2, we get:

$$r^2\frac{d\phi}{dr} = -w + r.\frac{dw}{dr}, \forall r \neq 0$$

Taking the derivative with respect to r and applying the chain rule of differentiation to the second term on the right:

$$\begin{aligned}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) &= \frac{d}{dr}\left(-w + r.\frac{dw}{dr}\right)\\ &= -\frac{dw}{dr} + r\frac{d^2w}{dr^2} + 1.\frac{dw}{dr}\\ &= r\frac{d^2w}{dr^2}\end{aligned}$$

Dividing by r^2 we get an expression for the Laplacian operator in term of the variable w:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = +\frac{1}{r}\frac{d^2w}{dr^2} \tag{18}$$

Substituting Eqn. 18 and 16 into Eqn. 15, we get:

$$\frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{L^2}\frac{w}{r} = 0$$

For all values of r not equal to zero, this equation reduces to a simple form for which we can readily find a solution:

$$\frac{d^2w}{dr^2} = +\frac{1}{L^2}w \tag{19}$$

This equation is analogous to the equation of the simple harmonic oscillator with the time variable replaced by the spatial radial variable, and a positive sign on the right hand side, and allows an exponential solution with two constants of integration as:

$$w(r) = Ae^{-\frac{r}{L}} + Be^{+\frac{r}{L}} \tag{20}$$

Using Eqn. 16, we can now get the solution for the flux as;

$$\phi(r) = \frac{w(r)}{r} = A\frac{e^{-\frac{r}{L}}}{r} + B\frac{e^{+\frac{r}{L}}}{r} \tag{21}$$

The second term leads to a solution involving an infinite value for the flux, which is not practically feasible since the flux must be finite in magnitude, implying that the constant B = 0, and:

$$\phi(r) = A\frac{e^{-\frac{r}{L}}}{r} \tag{22}$$

To determine the second constant of integration A, knowing that the solution does not apply at the origin r = 0, we surround the source by a sphere of radius r and integrate the current J(r) over the surface. As the radius of the sphere shrinks to zero, the current integrated over the surface tends to the strength of the source S:

$$\lim_{r\to 0}\left[4\pi r^2 J(r)\right] = S \tag{23}$$

Applying Fick's law from Eqn. 5 to the flux solution in Eqn. 22, using the chain law of differentiation, we get:

$$\begin{aligned} J(r) &= -D\nabla\phi(r) = -D\frac{d\phi(r)}{dr} = -D\frac{d}{dr}\left(A\frac{e^{-\frac{r}{L}}}{r}\right) \\ &= -DA\frac{d}{dr}\left(\frac{1}{r}.e^{-\frac{r}{L}}\right) = -DA\left(\frac{1}{r}e^{-\frac{r}{L}}(-\frac{1}{L}) - e^{-\frac{r}{L}}\frac{1}{r^2}\right) \\ &= DA\left(\frac{1}{Lr}e^{-\frac{r}{L}} + \frac{1}{r^2}e^{-\frac{r}{L}}\right) \end{aligned} \tag{24}$$

Substituting from Eqn. 24 into Eqn. 23 we get:

$$\lim_{r\to 0}\left[4\pi r^2 J(r)\right] = \lim_{r\to 0}\left[4\pi r^2 .DA\left(\frac{1}{Lr}e^{-\frac{r}{L}}+\frac{1}{r^2}e^{-\frac{r}{L}}+\right)\right] = S$$

$$= \lim_{r\to 0}\left[4\pi DA\left(\frac{r}{L}e^{-\frac{r}{L}}+e^{-\frac{r}{L}}\right)\right]$$

$$= S \qquad ,\forall r \neq 0.$$

Taking the limit yields zero for the first term and unity and unity for the second term on the right hand side, resulting in the value of the integration constant A:

$$4\pi DA.1 = S \;\Rightarrow A = \frac{S}{4\pi D} \tag{25}$$

Substituting A in the expression for the flux we finally get:

$$\phi(r) = \frac{Se^{-\frac{r}{L}}}{4\pi Dr} \tag{26}$$

It is of interest to compare this expression for the flux in a diffusing medium compared with the one for the flux in a vacuum: one of the radial position r factors in the denominator has been replaced by a decaying exponential in the numerator including the diffusion length L, and by the diffusion coefficient D in the denominator restoring the appropriate units to the expression for the flux. The solution is not valid at the origin r = 0, since r was cancelled on both sides of an equation during the derivation, which makes the solution valid everywhere except at the origin.

PROCEDURE TO ESTIMATE THE FLUX FROM A POINT SOURCE IN A DIFFUSING MEDIUM

The following procedure allows the display the results from Eqn. 26 requiring an input of the source strength S, diffusion coefficient D, and diffusion length L for different moderator media.

```
!       Neutron flux from a point source in an infinite nonmultiplying
!       moderator medium.
!       phi(r)=S*exp(-r/L)/4*Pi*D*r
!       Program saves output to file:output1
!       This output file can be exported to a plotting routine
!       M. Ragheb, Univ. of Illinois at Urbana-Champaign
!
        program flux
        real :: Pi = 3.14159
!       source strength
        real :: S = 10.0E+10
!       diffusion coefficient (D2O)
        real :: D = 0.62
```

```
!         diffusion length
          real :: L = 116.0.0 (D2O)
          integer :: steps=10
          real phi(100), r(100)
          write(*,*) S,D,L
!         Open output file
          open(10,file='output4')
!         Calculate ratio phi(r)
          do i =  1, steps
                    r(i) = i
                    phi(i) = (S*exp(-r(i)/L))/(4.0*Pi*D*r(i))
!         Write results on output file
                    write(10,*) r(i), phi(i)
!         Display results on screen
                    write(*,*) r(i), phi(i)
!                   pause
          end do
          end
```

Figure 4. Procedure for generating the flux from a point source in an infinite medium.

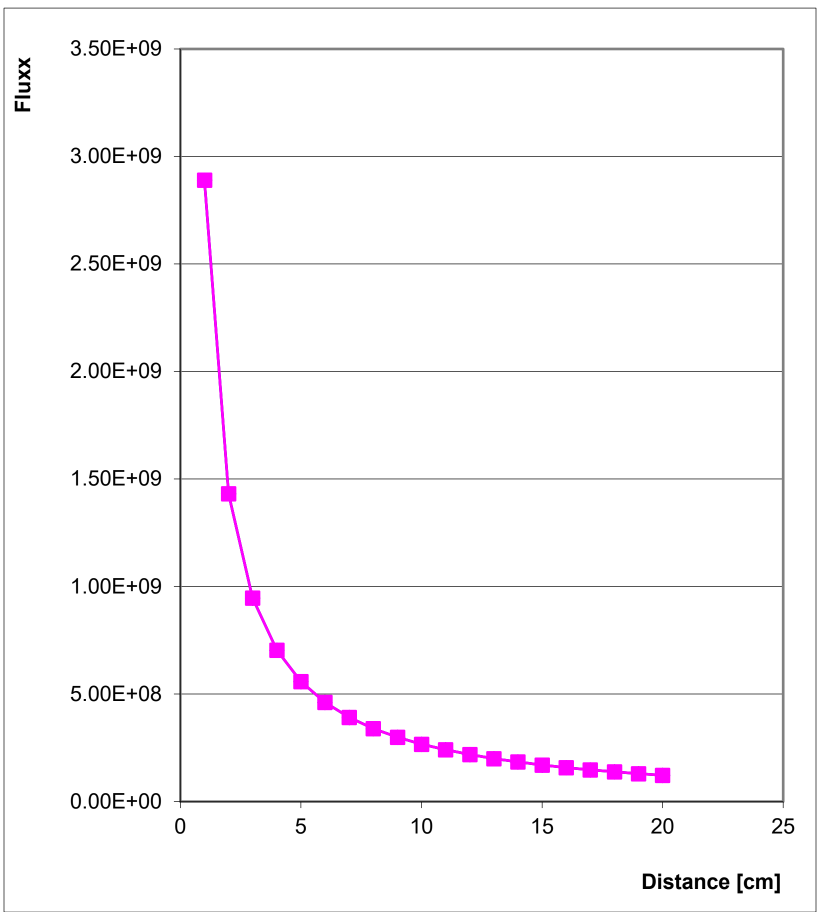

Figure 5. Neutron flux from a point source of strength $S=10^{10}$ [n/sec] in H_2O with a diffusion coefficient D = 0.164 [cm] and diffusion length L = 2.73 [cm].

TWO ENERGY GROUPS NEUTRON MODERATION

For a fast and thermal energy group of neutrons the moderating properties are different as shown in Table 1.

Table 1. Fast and thermal neutron ages and diffusion areas for different moderators.

Moderator	Fast group age τ [cm^2]	Thermal group diffusion length L^2 [cm^2]
H_2O	27	8.1
D_2O	131	3x10^4
Be	102	480
Graphite	368	3,500

One can write two diffusion equations, one for the fast group, and one for the thermal group.

$$\begin{aligned} Fast: \quad & D_{fast}\nabla^2\phi_{fast} - \Sigma_{fast}\phi_{fast} + S = 0 \\ Thermal: \quad & D_{th}\nabla^2\phi_{th} - \Sigma_{ath}\,\phi_{th} + \Sigma_{fast}\phi_{fast} = 0 \end{aligned} \tag{27}$$

The slowing down density q is a loss term in the fast group equation, but a gain term in the thermal group equation coupling the two groups:

$$q = \Sigma_{fast}\,\phi_{fast} \tag{28}$$

Dividing the fast and thermal group equations by their respective diffusion coefficients:

$$\begin{aligned} &\nabla^2\phi_{fast} - \frac{\Sigma_{fast}}{D_{fast}}\phi_{fast} + \frac{S}{D_{fast}} = 0 \\ &\nabla^2\phi_{th} - \frac{\Sigma_{ath}}{D_{th}}\phi_{th} + \frac{\Sigma_{fast}}{D_{th}}\phi_{fast} = 0 \end{aligned} \tag{29}$$

Defining the fast neutron age as:

$$\tau = \frac{D_{fast}}{\Sigma_{fast}} \tag{30}$$

and the thermal group diffusion area as:

$$L^2 = \frac{D_{th}}{\Sigma_{ath}} \tag{31}$$

we can rewrite the equations as:

$$\begin{aligned} \nabla^2 \phi_{fast} - \frac{1}{\tau}\phi_{fast} &= -\frac{S}{D_{fast}} \\ \nabla^2 \phi_{th} - \frac{1}{L^2}\phi_{th} &= -\frac{\Sigma_{fast}}{D_{th}}\phi_{fast} \end{aligned} \tag{32}$$

In spherical geometry the fast group flux has a solution similar to that of Eqn. 26, yielding:

$$\phi_{fast}(r) = \frac{Se^{-\frac{r}{\sqrt{\tau}}}}{4\pi D_{fast} r} \tag{33}$$

Substituting the value for the fast flux in the second equation, one can solve the inhomogeneous equation:

$$\nabla^2 \phi_{th} - \frac{1}{L^2}\phi_{th} = -\frac{\Sigma_{fast}}{D_{th}} \frac{Se^{-\frac{r}{\sqrt{\tau}}}}{4\pi D_{fast} r} \tag{34}$$

to yield the thermal group flux as:

$$\phi_{th}(r) = \frac{SL^2}{4\pi D_{th}(L^2 - \tau)r}\left(e^{-\frac{r}{L}} - e^{-\frac{r}{\sqrt{\tau}}}\right) \tag{35}$$

FINITE MEDIA DIFFUSION

We consider a planar fuel unit cell in a fission reactor surrounded with a moderator with a constant neutron source S. Our goal is to estimate the fluxes in the two different media of moderator and fuel. Such calculations are important for the estimation of the fuel utilization factor f in the four factor formula.

We define a source of thermal neutrons in the moderator region alone:

$$\begin{aligned} S(x) &= 0 \qquad -a \le x < 0 \\ &= S_0 \qquad 0 \le x \le b \end{aligned} \tag{36}$$

In the fuel region, the neutron diffusion equation in a one dimensional cartesian coordinate system is:

$$\nabla^2 \phi_{fuel}(x) = +\frac{1}{L_{fuel}^2}\phi_{fuel}(x) \tag{37}$$

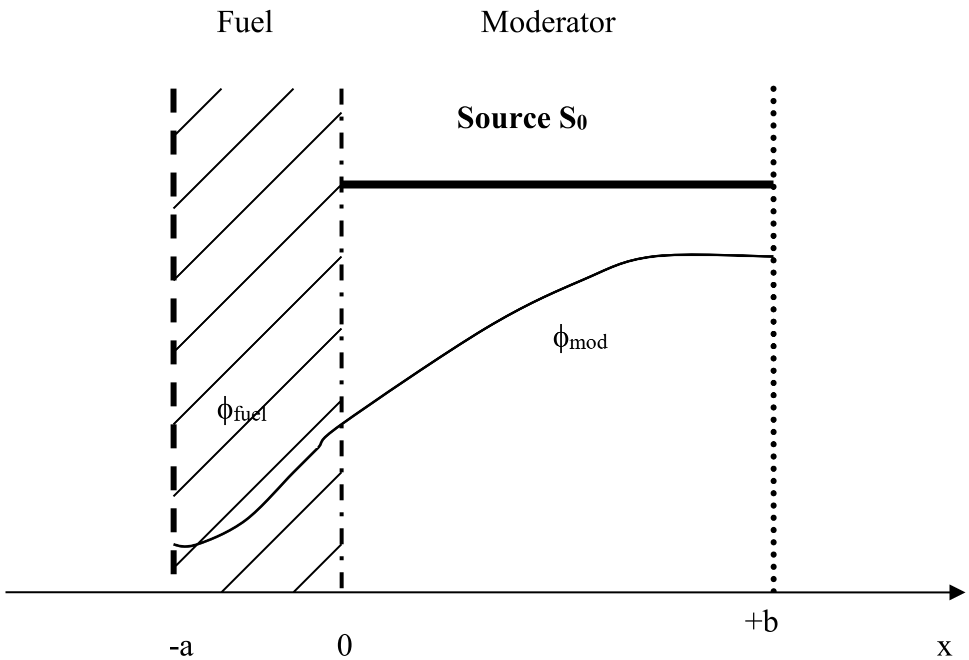

Figure 6. Unit cell of plate reactor fuel surrounded by a moderator.

For a finite medium, we chose the hyperbolic cosine solution to the equation instead of the exponential form, since it is more suitable for finite size media in terms of application of the boundary conditions:

$$\phi_{fuel}(x) = C\cosh\frac{(x+a)}{L_{fuel}} + E\sinh\frac{(x+a)}{L_{fuel}} \tag{38}$$

To apply the boundary conditions and estimate the constants C and E, we write the expression for the current:

$$\begin{aligned} J(x) &= -D_{fuel}\frac{d\phi_{fuel}}{dx} \\ &= -C\frac{D_{fuel}}{L_{fuel}}\sinh\frac{(x+a)}{L_{fuel}} - E\frac{D_{fuel}}{L_{fuel}}\cosh\frac{(x+a)}{L_{fuel}} \end{aligned} \tag{39}$$

Based on symmetry considerations, the current at x = -a should be zero:

$$J(-a) = -C\frac{D_{fuel}}{L_{fuel}}\sinh\frac{(-a+a)}{L_{fuel}} - E\frac{D_{fuel}}{L_{fuel}}\cosh\frac{(-a+a)}{L_{fuel}}$$
$$= -E\frac{D_{fuel}}{L_{fuel}} = 0 \qquad \Rightarrow E = 0$$

The cosh solution is retained, and the sinh solution is dropped (E = 0), yielding:

$$\phi_{fuel}(x) = C\cosh\frac{(x+a)}{L_{fuel}} \tag{40}$$

In the moderator region, the diffusion equation with the source S_0 is:

$$\nabla^2\phi_{\text{mod}}(x) = +\frac{1}{L^2_{\text{mod}}}\phi_{\text{mod}}(x) - \frac{S_0}{D_{\text{mod}}} \tag{41}$$

The particular solution is:

$$\begin{aligned} \phi_{\text{mod}\,p} &= \frac{S_0}{D_{\text{mod}}}L^2_{\text{mod}} = \frac{S_0}{\Sigma_{a\,\text{mod}}} \\ where: \frac{\Sigma_{a\,\text{mod}}}{D_{\text{mod}}} &= \frac{1}{L^2_{\text{mod}}} \end{aligned} \tag{42}$$

The complementary solution is:

$$\phi_{\text{mod}\,c}(x) = -A\cosh\frac{(b-x)}{L_{\text{mod}}} - B\sinh\frac{(b-x)}{L_{\text{mod}}} \tag{43}$$

Adding the complementary and particular solutions, we get:

$$\begin{aligned} \phi_{\text{mod}}(x) &= \phi_{\text{mod}\,c}(x) + \phi_{\text{mod}\,p}(x) \\ &= -A\cosh\frac{(b-x)}{L_{\text{mod}}} - B\sinh\frac{(b-x)}{L_{\text{mod}}} + \frac{S_0}{\Sigma_{a\,\text{mod}}} \end{aligned} \tag{44}$$

Estimating the current in the moderator region yields:

$$J(x) = -A\frac{D_{\text{mod}}}{L_{\text{mod}}}\sinh\frac{(b-x)}{L_{\text{mod}}} - B\frac{D_{\text{mod}}}{L_{\text{mod}}}\cosh\frac{(b-x)}{L_{\text{mod}}} \tag{45}$$

Applying the boundary condition:

$$
\begin{aligned}
J(b) &= -A\frac{D_{\text{mod}}}{L_{\text{mod}}}\sinh\frac{(b-b)}{L_{\text{mod}}} - B\frac{D_{\text{mod}}}{L_{\text{mod}}}\cosh\frac{(b-b)}{L_{\text{mod}}} \\
&= -B\frac{D_{\text{mod}}}{L_{\text{mod}}} = 0 \qquad \Rightarrow B = 0
\end{aligned}
\tag{46}
$$

Again, the cosh solution is retained, and the sinh solution dropped (B=0), yielding:

$$
\phi_{\text{mod}}(x) = -A\cosh\frac{(b-x)}{L_{\text{mod}}} + \frac{S_0}{\Sigma_{a\,\text{mod}}} \tag{47}
$$

Further applying the two interface boundary conditions of the continuity for the flux and the current at x = 0, yields the two equations in the two unknowns A and C:

Flux continuity:

$$
\begin{aligned}
&\phi_{\text{fuel}} = \phi_{\text{mod}} \\
&C\cosh\frac{a}{L_{fuel}} = \frac{S_0}{\Sigma_{a\,\text{mod}}} - A\cosh\frac{b}{L_{\text{mod}}}
\end{aligned}
\tag{48}
$$

Current continuity:

$$
\begin{aligned}
&J_{\text{fuel}} = J_{\text{mod}} \\
&-C\frac{D_{fuel}}{L_{fuel}}\sinh\frac{a}{L_{fuel}} = -A\frac{D_{\text{mod}}}{L_{\text{mod}}}\sinh\frac{b}{L_{\text{mod}}}
\end{aligned}
\tag{49}
$$

Writing the two linear equations in the two unknowns A and C:

$$
\begin{aligned}
&A\cosh\frac{b}{L_{\text{mod}}} \qquad + C\cosh\frac{a}{L_{fuel}} \qquad = \frac{S_0}{\Sigma_{a\,\text{mod}}} \\
&A\frac{D_{\text{mod}}}{L_{\text{mod}}}\sinh\frac{b}{L_{\text{mod}}} - C\frac{D_{fuel}}{L_{fuel}}\sinh\frac{a}{L_{fuel}} = 0
\end{aligned}
\tag{50}
$$

or:

$$
\begin{pmatrix}
\cosh\frac{b}{L_{\text{mod}}} & \cosh\frac{a}{L_{fuel}} \\
\frac{D_{\text{mod}}}{L_{\text{mod}}}\sinh\frac{b}{L_{\text{mod}}} & -\frac{D_{fuel}}{L_{fuel}}\sinh\frac{a}{L_{fuel}}
\end{pmatrix}
\begin{pmatrix} A \\ B \end{pmatrix}
=
\begin{pmatrix} \frac{S_0}{\Sigma_{a\,\text{mod}}} \\ 0 \end{pmatrix}
\tag{51}
$$

The solution for the two constants A and C are:

$$A=\frac{-\frac{S_0}{\Sigma_{a\,\mathrm{mod}}}\frac{D_{fuel}}{L_{fuel}}\sinh\frac{a}{L_{fuel}}}{-\frac{D_{fuel}}{L_{fuel}}\sinh\frac{a}{L_{fuel}}\cosh\frac{b}{L_{\mathrm{mod}}}-\frac{D_{\mathrm{mod}}}{L_{\mathrm{mod}}}\sinh\frac{b}{L_{\mathrm{mod}}}\cosh\frac{a}{L_{fuel}}}$$

$$C=\frac{-\frac{S_0}{\Sigma_{a\,\mathrm{mod}}}\frac{D_{\mathrm{mod}}}{L_{\mathrm{mod}}}\sinh\frac{b}{L_{\mathrm{mod}}}}{-\frac{D_{fuel}}{L_{fuel}}\sinh\frac{a}{L_{fuel}}\cosh\frac{b}{L_{\mathrm{mod}}}-\frac{D_{\mathrm{mod}}}{L_{\mathrm{mod}}}\sinh\frac{b}{L_{\mathrm{mod}}}\cosh\frac{a}{L_{fuel}}}$$

Multiplying the expressions for A and C by $L_{fuel}L_{mod}$, simplifies them to:

$$A=\frac{S_0}{\Sigma_{a\,\mathrm{mod}}}\frac{L_{\mathrm{mod}}D_{fuel}\sinh\frac{a}{L_{fuel}}}{L_{\mathrm{mod}}D_{fuel}\sinh\frac{a}{L_{fuel}}\cosh\frac{b}{L_{\mathrm{mod}}}+L_{fuel}D_{\mathrm{mod}}\sinh\frac{b}{L_{\mathrm{mod}}}\cosh\frac{a}{L_{fuel}}}$$

$$C=\frac{S_0}{\Sigma_{a\,\mathrm{mod}}}\frac{L_{fuel}D_{\mathrm{mod}}\sinh\frac{b}{L_{\mathrm{mod}}}}{L_{\mathrm{mod}}D_{fuel}\sinh\frac{a}{L_{fuel}}\cosh\frac{b}{L_{\mathrm{mod}}}+L_{fuel}D_{\mathrm{mod}}\sinh\frac{b}{L_{\mathrm{mod}}}\cosh\frac{a}{L_{fuel}}} \quad (52)$$

Thus the solutions for the fluxes in the fuel and in the moderator regions are:

$$\phi_{fuel}(x)=C\cosh\frac{(x+a)}{L_{fuel}}$$

$$=\frac{S_0}{\Sigma_{a\,\mathrm{mod}}}\frac{L_{fuel}D_{\mathrm{mod}}\sinh\frac{b}{L_{\mathrm{mod}}}.\cosh\frac{(x+a)}{L_{fuel}}}{L_{\mathrm{mod}}D_{fuel}\sinh\frac{a}{L_{fuel}}\cosh\frac{b}{L_{\mathrm{mod}}}+L_{fuel}D_{\mathrm{mod}}\sinh\frac{b}{L_{\mathrm{mod}}}\cosh\frac{a}{L_{fuel}}} \quad (53)$$

$$\forall -a\le x\le 0$$

$$\phi_{\mathrm{mod}}(x)=\frac{S_0}{\Sigma_{a\,\mathrm{mod}}}-A\cosh\frac{(b-x)}{L_{\mathrm{mod}}}$$

$$=\frac{S_0}{\Sigma_{a\,\mathrm{mod}}}\left[1-\frac{L_{\mathrm{mod}}D_{fuel}\sinh\frac{a}{L_{fuel}}\cosh\frac{(b-x)}{L_{\mathrm{mod}}}}{L_{\mathrm{mod}}D_{fuel}\sinh\frac{a}{L_{fuel}}\cosh\frac{b}{L_{\mathrm{mod}}}+L_{fuel}D_{\mathrm{mod}}\sinh\frac{b}{L_{\mathrm{mod}}}\cosh\frac{a}{L_{fuel}}}\right] \quad (54)$$

$$\forall\ 0\le x\le b.$$

APPENDIX

THE SIMPLE HARMONIC OSCILLATOR (SHO)

The solutions to the differential equations describing the Simple Harmonic Oscillator (SHO) are analogous to the equations describing the diffusion of neutrons in non-multiplying or multiplying media. In the analogy, the time variable in the SHO equations and solutions is replaced by the spatial variable in the case of neutron diffusion.

HORIZONTAL SPRING SIMPLE HARMONIC OSCILLATOR

Writing the equation of motion for a mass m on an horizontal plane with a viscous damping coefficient β, held by a spring with constant k:

$$m\ddot{x} = -\beta\dot{x} - kx \tag{1}$$

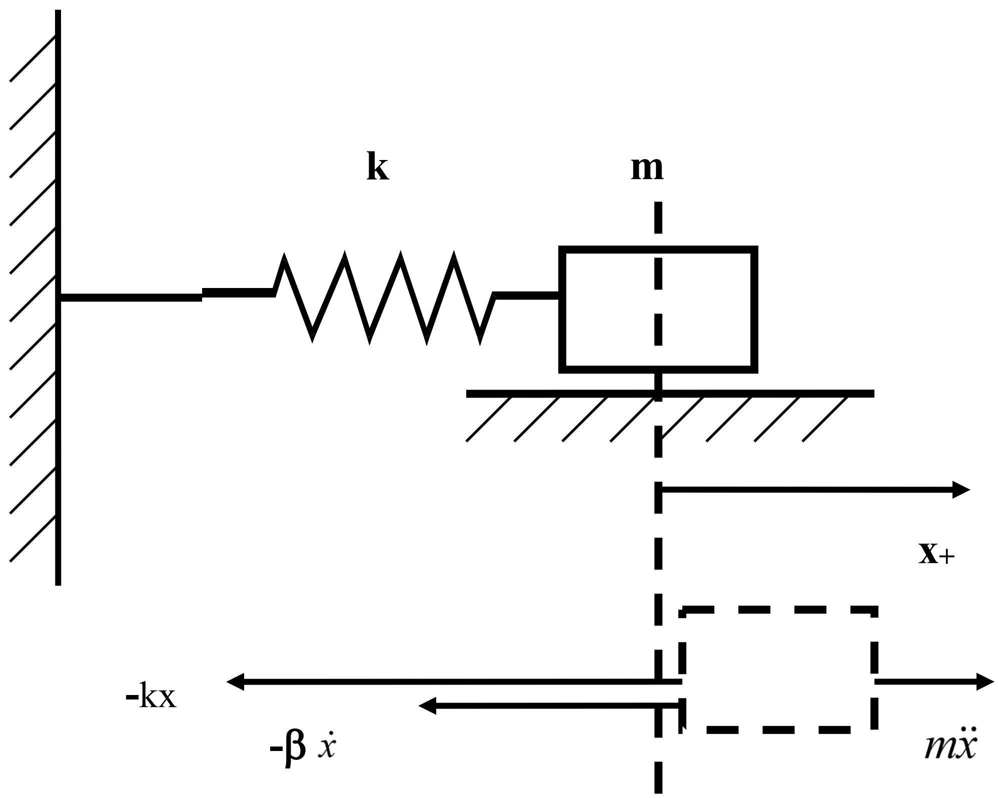

Figure A1. Horizontal spring simple harmonic oscillator.

Ignoring the viscous damping term, we get:

$$m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m}x$$

or:

$$\ddot{x} = -\omega^2 x \qquad (2)$$

where:

$$\text{frequency } \omega = \sqrt{\frac{k}{m}} = 2\pi f$$

This is analogous to the harmonic motion of the simple pendulum where the energy oscillates between being in the form of kinetic energy of the mass m and potential energy. It also possesses an electrical circuit analog.

TUNING CIRCUIT ANALOG

If we consider a tuning circuit composed of a resistor R, an inductance L, and a capacitance C, the voltage through the circuit in terms of the charge q is given by:

$$V(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} \qquad (3)$$

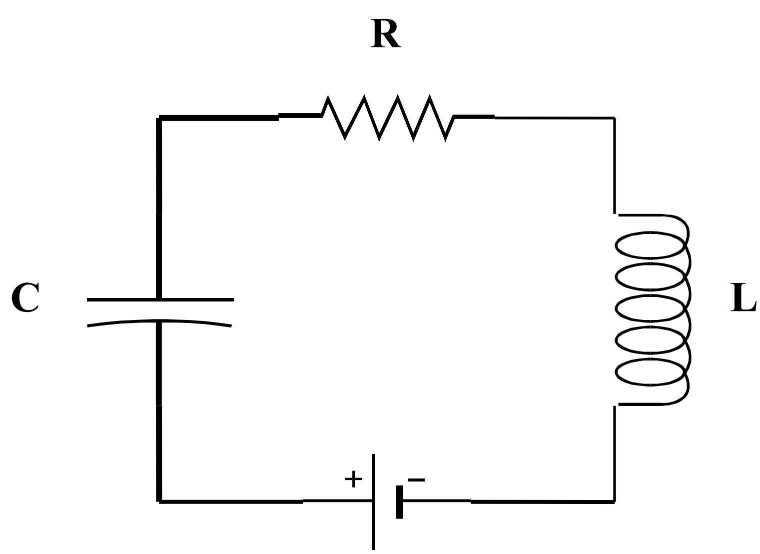

Figure A2. RLC tuning circuit with voltage source.

For no applied potential and zero dissipative resistive component R it reduces to:

$$L\frac{d^2q}{dt^2} = -\frac{q}{C}$$

or:

$$\frac{d^2q}{dt^2} = -\frac{1}{LC}q \qquad (4)$$

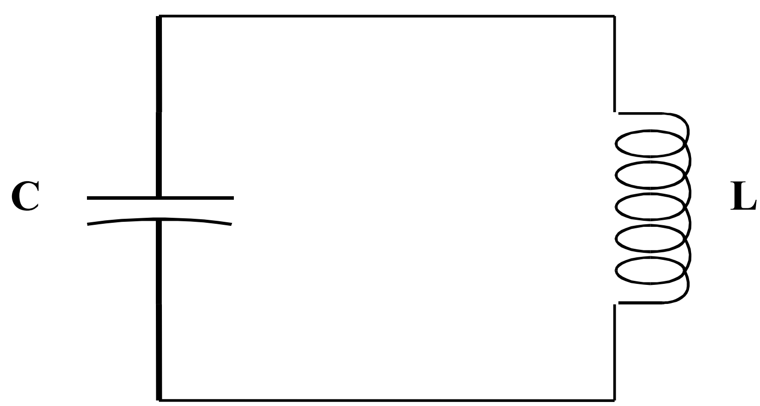

Figure A3. The LC oscillator circuit without resistive dissipative element. The energy stored in the electric field in the capacitor and the energy stored in the magnetic field in the inductor oscillate between one form to the other.

This can be written as:

$$\ddot{q} = -\omega^2 q \qquad (5)$$

where:

frequency $\omega = \frac{1}{\sqrt{LC}} = 2\pi f$

OSCILLATORY HARMONIC SOLUTION

Both Eqns. 2 and 5 have the general form:

$$\ddot{x}(t) = -\omega^2 x(t) \qquad (6)$$

They possess an oscillatory or harmonic solution which can be written in either one of the alternative forms:

$$
\begin{aligned}
x(t) &= A\cos(\omega t) + B\sin(\omega t) \\
x(t) &= A^{'}e^{+jwt} \qquad + B^{'}e^{-jwt}
\end{aligned}
\tag{7}
$$

where: $j = \sqrt{-1}$

since:

$$
\begin{aligned}
e^{+j\omega t} &= \cos(\omega t) + j\sin(\omega t) \\
e^{-j\omega t} &= \cos(\omega t) - j\sin(\omega t)
\end{aligned}
\tag{8}
$$

or:

$$
\begin{aligned}
\cos(\omega t) &= \frac{e^{+j\omega t} + e^{-j\omega t}}{2} \\
\sin(\omega t) &= \frac{e^{+j\omega t} - e^{-j\omega t}}{2j}
\end{aligned}
\tag{9}
$$

EXPONENTIAL GROWTH AND DECAY SOLUTION

Whenever the right hand side of Eqn. 6 has a positive rather than a negative sign:

$$
\ddot{x}(t) = +\omega^2 x(t) \tag{10}
$$

The solution is a negative and positive exponential decay and growth given by:

$$
\begin{aligned}
x(t) &= A^{''}\cosh(\omega t) + B^{''}\sinh(\omega t) \\
x(t) &= A^{'''}e^{+wt} \qquad + B^{'''}e^{-wt}
\end{aligned}
\tag{11}
$$

where cosh(x) and sinh(x) are the hyperbolic cosine and sine functions, with:

$$
\begin{aligned}
e^{+\omega t} &= \cosh(\omega t) + \sinh(\omega t) \\
e^{-\omega t} &= \cosh(\omega t) - \sinh(\omega t)
\end{aligned}
\tag{12}
$$

or:

$$\cosh(\omega t) = \frac{e^{+\omega t} + e^{-\omega t}}{2}$$
$$\sinh(\omega t) = \frac{e^{+\omega t} - e^{-\omega t}}{2} \qquad (13)$$

The exponential form of the solution is usually adopted in the case of infinite media, whereas the hyperbolic sine and cosine form is used in the case of finite media, facilitating the determination of the constants of integration in either case.

EXERCISES

1. Calculate the thickness of a shield made out of:
a) Water.
b) Graphite.
that would attenuate a beam of neutrons by a factor of:
a) One million times (10^{-6}).
b) One billion times (10^{-9}).
2. Plot the neutron fluxes away from the origin for a point neutron source of strength $S=10^{10}$ [n/sec] in an infinite medium of the following moderators:
a) H_2O: diffusion coefficient D = 0.164 cm, diffusion length L = 2.73 cm.
b) D_2O: diffusion coefficient D = 0.620 cm, diffusion length L = 116.0 cm.
3. Compare the fluxes generated in the previous problem to that in a vacuum.
4. Estimate the flux and current in vacuum at the midpoint between two sources of strength S each and separated by a distance of 100 cm.
Hint: The current is a vector and adds up vectorially, whereas the flux is a scalar.
5. Estimate the flux and current in a diffusing medium with diffusion length L = 2.73 cm and diffusion coefficient D = 0.164 cm (H_2O) at the midpoint between two sources of strength S each and separated by a distance of 100 cm.
6. Estimate the flux and current in vacuum at the center of:
a) A square,
b) A cube,
c) An equilateral triangle
with a source S at each corner and a side length of 100 cm.
7. Estimate the flux and current in a diffusing medium with diffusion length L = 2.73 cm and diffusion coefficient D = 0.164 cm (H_2O) in the form of:
a) A square,
b) A cube,
c) An equilateral triangle
with a source S at each corner and a side length of 100 cm.
8. Through direct substitution prove that the different general forms given for the solution of the Simple Harmonic Oscillator do indeed satisfy the underlying differential equation.
9. Compare the magnitude of the neutron flux generated by a source of strength $S=10^{10}$ [neutrons / second], at a distance of 100 cm from the source in spherical geometry,
1. In a vacuum.

2. In a diffusing medium with diffusion coefficient D=1 cm, and macroscopic absorption cross section equal to 0.1 cm^{-1}.

10. Prove that the thermal group flux for a two group slowing down of a fast neutron source S in a moderating medium is given by:

$$\phi_{th}(r) = \frac{SL^2}{4\pi D_{th}(L^2 - \tau)r}\left(e^{-\frac{r}{L}} - e^{-\frac{r}{\sqrt{\tau}}}\right)$$